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Abstract

One of the most important aspects of the (statistical) analysis of imprecise data is the usage of a suitable distance on the family of all compact, convex fuzzy sets, which is not too hard to calculate and which reflects the intuitive meaning of fuzzy sets. On the basis of expressing the metric of Bertoluzza \textit{et al}. (1995) in terms of the mid points and spreads of the corresponding intervals we construct new families of metrics on the family of all $d$-dimensional compact convex sets as well as on the family of all $d$-dimensional compact convex fuzzy sets. It is shown that these metrics not only fulfill many good properties, but also that they are easy to calculate and easy to manage for statistical purposes, and therefore useful from the practical point of view.

Key words: distance, generalized mid and spread, convex compact sets, imprecise data analysis, fuzzy sets, support functions

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1 Introduction

The concept of fuzzy sets is more and more used in order to model non-random uncertainty or imprecision that typically arises in the context of collecting or processing different kinds of realistic data. In fact, every measurement of a continuous physical quantity is imprecise - the imprecision may for instance be caused by the fact that the measurement device rounds to a certain number of digits or may be due to physical conditions (Heisenberg uncertainty in the microscopic level, etc.). On the other hand, in qualifying and classifying individuals one makes often use of judgements or linguistic labels which cannot be properly identified with exact real or vectorial values, but they can be easily described in terms of fuzzy values. Consequently fuzzy sets have been applied in data analysis in various areas like forestry, structural analysis, hydrology and economics [1,3,4,17].

In the one-dimensional setting fuzzy sets with non-empty compact intervals as $\alpha$-cuts are usually considered, since they are the most natural generalization of intervals. In order to define a reasonable metric on the family $\mathcal{F}_c(\mathbb{R})$ of such fuzzy sets one can make use of any metric $\delta$ on the family $\mathcal{K}_c(\mathbb{R})$ of all non-empty compact intervals, apply this metric to the family of all corresponding $\alpha$-cuts and integrate these distances as function of $\alpha$ (with respect to a probability measure having support $[0,1]$).

Bertoluzza et al. [2] remarked that the utilization of the Hausdorff metric $\delta_H$, as well as the $\rho_2$ metric [6] on $\mathcal{K}_c(\mathbb{R})$ is not recommendable, because both metrics only take into account distances between the infima and the suprema of the considered intervals. Therefore Bertoluzza et al. [2] essentially proposed a weighted average of the distances between all the convex combinations of the infima and suprema as metric $d_W$ in $\mathcal{K}_c(\mathbb{R})$. After this, they defined a metric $D^\varphi_W$ in $\mathcal{F}_c(\mathbb{R})$ as weighted average of the distances of the level sets. In [2] and [10] it is shown that the $D^\varphi_W$-metric has a suitable intuitive meaning, in addition to very good operational properties. The usefulness of this metric is mainly due to two facts. Firstly, it involves not only distances between extreme points (infima and suprema), but also distances between inner points in the intervals (except for some particular choices of $W$ that will be mentioned later). Secondly, it is an $L_2$-type metric, which implies very good properties in connection with least squares methods and other optimization problems. For this reason, the $D^\varphi_W$-metric has been widely used in statistical problems [12,8,16].

We will show that the expression of $d_W$ in terms of mids and spreads can easily be extended to the multidimensional case, and that this construction yields a new class of metrics that both exhibit many good and intuitive properties and that, moreover, are easy to calculate.
The rest of the paper is organized as follows. In the next section, the notations that will be used in the sequel are gathered and some important definitions are stated. Section 3 presents the basic construction given in [2] and mentions some additional properties that are useful in the multidimensional setting. Afterwards in Section 4 the generalization of the mids and spreads are introduced and some useful properties are established. On this basis, the metric $d_W$ is extended to the $d$-dimensional setting $K_c(\mathbb{R}^d)$, the most important properties of the obtained metrics are summarized and it is shown that there are some other natural and intuitive alternative expressions useful for theoretical developments. Furthermore the connection with the usual Hausdorff metric $\delta_H$ on $K_c(\mathbb{R}^d)$ from the topological point of view will be analyzed and some examples will be stated in order to show that the good properties of $d_W$ are also fulfilled in the multidimensional setting. Section 5 presents the extension of the obtained metrics from $K_c(\mathbb{R}^d)$ to $F_c(\mathbb{R}^d)$, and shows the interrelation between the new metrics and other commonly used metrics on $F_c(\mathbb{R}^d)$. Finally, in Section 6 we make some concluding remarks and state open problems.

## 2 Preliminaries

Throughout the whole paper $K_c(\mathbb{R})$ denotes the family of all non-empty compact intervals and $K_c(\mathbb{R}^d)$ denotes the family of all $d$-dimensional non-empty compact convex sets in $\mathbb{R}^d$. The Hausdorff metric $\delta_H$ on $K_c(\mathbb{R}^d)$ is defined by

$$\delta_H(A, B) = \max \left\{ \max_{a \in A} \min_{b \in B} |a - b|, \max_{b \in B} \min_{a \in A} |a - b| \right\}, \quad (1)$$

whereby $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^d$. It is well known that every non-empty compact convex set $A \in K_c(\mathbb{R}^d)$ can be characterized by its so-called support function $s_A(\cdot)$ [19,20], defined by

$$s_A(u) = \sup_{a \in A} \langle a, u \rangle \quad \text{for every} \quad u \in S^{d-1}, \quad (2)$$

whereby $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^d$ and $S^{d-1}$ denotes the unit sphere in $\mathbb{R}^d$. Using support functions another type of metrics on $K_c(\mathbb{R}^d)$ (the well-known $L_p$-metrics going back to Vitale [20]) can be defined as follows:

$$\rho_p(A, B) = \left( \int_{S^{d-1}} |s_A(u) - s_B(u)|^p \, d\vartheta_d(u) \right)^{\frac{1}{p}} \quad (3)$$

Thereby $p \geq 1$ and $\vartheta_d$ denotes the normalized Lebesgue (surface) measure on $S^{d-1}$. For the most important properties of $\rho_p$ see [20].

The family of all $d$-dimensional fuzzy sets $F_c(\mathbb{R}^d)$ considered throughout this
paper is defined by

\[ F_c(\mathbb{R}^d) = \left\{ \tilde{A} : \mathbb{R}^d \to [0, 1] \mid \tilde{A}_\alpha \in K_c(\mathbb{R}^d) \text{ for every } \alpha \in [0, 1] \right\}, \]  

whereby the \( \alpha \)-cut \( \tilde{A}_\alpha \) is defined by \( \tilde{A}_\alpha = \{ x \in \mathbb{R}^d \mid \tilde{A}(x) \geq \alpha \} \) for every \( \alpha \in (0, 1] \) and the set \( A_0 \), called the support of \( A \), is defined as the (topological) closure of \( \bigcup_{\alpha > 0} A_\alpha \).

Every fuzzy set \( \tilde{A} \in F_c(\mathbb{R}^d) \) can also be characterized by means of its support function \( s_{\tilde{A}} \), defined by

\[ s_{\tilde{A}}(u, \alpha) = s_{\tilde{A}_\alpha}(u) = \sup_{a \in \tilde{A}_\alpha} \langle u, a \rangle \]  

for every \( u \in S^{d-1} \) and \( \alpha \in (0, 1] \) [14,15].

Applying Zadeh’s extension principle [21] for the sum and the product by scalar (or, equivalently, \( \alpha \)-cut-wise Minkowski sum and scalar multiplication) the family \( F_c(\mathbb{R}^d) \) can be regarded as a semi-linear space [6]. Furthermore, using the concept of support functions for fuzzy sets, the family \( F_c(\mathbb{R}^d) \) can also be seen as a convex cone in the Hilbert space \( H = L^2(S^{d-1} \times [0, 1], \vartheta_d \otimes \lambda) \) (\( \lambda \) denoting the Lebesgue measure on \( [0, 1] \)). Actually the mapping \( \Phi(\tilde{A}) = s_{\tilde{A}} \) establishes an embedding of \( F_c(\mathbb{R}^d) \) in \( H \). This embedding is an isometry if \( F_c(\mathbb{R}^d) \) is endowed with the metric \( \rho_2 \), defined by

\[ \rho_2(\tilde{A}, \tilde{B}) = \left( \int_{[0,1]} \int_{S^{d-1}} |s_{\tilde{A}}(u, \alpha) - s_{\tilde{B}}(u, \alpha)|^2 d\vartheta_d(v) d\lambda(\alpha) \right)^{1/2} \]  

for every pair \( \tilde{A}, \tilde{B} \in F_c(\mathbb{R}^d) \). For the most important properties of the metric space \( (F_c(\mathbb{R}^d), \rho_2) \) and other metrics on \( F_c(\mathbb{R}^d) \) see [15].

### 3 One-dimensional setting

In the sequel the set of all (Borel) probability measures \( W \) on \([0, 1]\), which are not concentrated on one single point will be denoted by \( P_0 \). For every \( W \in P_0 \) the Bertoluzza metric \( d_W \) on \( K_c(\mathbb{R}) \) according to [2] is given by

\[ d_W(A, B) = d_W^2([a, \pi], [b, \bar{b}]) = \int_{[0,1]} \left( t(a - b) + (1 - t)(\pi - \bar{b}) \right)^2 dW(t) \]  

for every pair \( A = [a, \pi], B = [b, \bar{b}] \in K_c(\mathbb{R}) \).
The metric $d^2_W$ can also be expressed as

$$d^2_W(A, B) = (\text{mid } A - \text{mid } B)^2 + (\text{spr } A - \text{spr } B)^2 \int_{[0,1]} (2t - 1)^2 dW(t)$$

$$-2(\text{mid } A - \text{mid } B)(\text{spr } A - \text{spr } B) \int_{[0,1]} (2t - 1) dW(t),$$

where $\text{mid } A = (\bar{a} + a)/2$ is the centre of the interval $A$ and $\text{spr } A = (\bar{a} - a)/2$ is half of the length of $A$.

In (7) we observe that only the first and the second moment of $W$ are involved in the calculation of the distance.

**Remark 3.1** Every meaningful metric $m$ on $\mathcal{K}_c(\mathbb{R})$ should be invariant to rigid motions, i.e. $m(A + r, B + r) = m(A, B)$ and $m(-A, -B) = m(A, B)$ should hold for all $A, B \in \mathcal{K}_c(\mathbb{R})$ and every $r \in \mathbb{R}$.

From (7) it follows immediately that $d_W(A + r, B + r) = d_W(A, B)$ holds for all $A, B \in \mathcal{K}_c(\mathbb{R})$ and $r \in \mathbb{R}$. However, taking into account (8) we have that

$$d_W(-A, -B) = d_W(A, B)$$

for all $A, B \in \mathcal{K}_c(\mathbb{R})$ if and only if

$$\int_{[0,1]} (2t - 1) dW(t) = 0.$$ 

That is, if and only if $W$ is a measure with first moment $1/2$ (i.e. $\int_{[0,1]} t dW(t) = 1/2$).

According to Remark 3.1, we will denote by $\mathcal{P}_0^*$ the set of all measures $W \in \mathcal{P}_0$ having $1/2$ as first moment. Thus if $W \in \mathcal{P}_0^*$, we have that

$$d^2_W(A, B) = (\text{mid } A - \text{mid } B)^2 + (\text{spr } A - \text{spr } B)^2 \int_{[0,1]} (2t - 1)^2 dW(t).$$

Consequently, $W$ determines the relative importance of the squared distance between the spreads in relationship with the squared distance between the mids through $\theta_W = \int_{[0,1]} (2t - 1)^2 dW(t)$, which allows us to express the $d_W$ distance as

$$d^2_W(A, B) = (\text{mid } A - \text{mid } B)^2 + \theta_W (\text{spr } A - \text{spr } B)^2.$$ 

(9)

In addition we have that $0 < \theta_W \leq 1$ for all $W \in \mathcal{P}_0^*$, and conversely for all $\theta \in (0, 1]$, there exists $W \in \mathcal{P}_0^*$ such that $\theta_W = \theta$. To be precise, it is enough to consider measures giving weights $(\theta/2, 1 - \theta, \theta/2)$ to the points $(0, 1/2, 1)$.

Consequently, in order to choose a distance from the family $d_W$ in practice it suffices to fix the relative importance $\theta_W$ of the spreads against the mids.
If $\theta_W = 1$ then the measure $W$ is concentrated on $\{0, 1\}$. Thus, the metric only considers the distance of the boundaries of the corresponding intervals and we get the well-known $\rho_2$ metric [20,6], that is,

$$\rho_2^2(A, B) = \frac{1}{2}(b - a)^2 + \frac{1}{2}(b - a)^2 = \left(\text{mid } A - \text{mid } B\right)^2 + \left(\text{spr } A - \text{spr } B\right)^2$$

On the other hand, the relevance of the mid-spread representation is also stressed by the possibility of expressing the Hausdorff distance as

$$\delta_H(A, B) = \max\{|\bar{b} - \bar{a}|, |\bar{b} - \bar{a}|\} = |\text{mid } A - \text{mid } B| + |\text{spr } A - \text{spr } B|.$$ 

**Remark 3.2** According to equation (9) the weight of the distance of the mids is strictly greater than or equal to that of the spreads - this is in accordance with the fact that it seems natural that the distance of the mids is more important than that of the spreads since the mid, loosely speaking, determines the position of the set.

The following example underlines the advantage of the metric $d_W$ over $\delta_H$ and $\rho_2$ in the case that $W$ is, for instance, the Lebesque measure $\lambda$, for which $\theta_\lambda = 1/3$.

**Example 3.1** The $\delta_H$ and $\rho_2$ distances of for instance $[-2, 2]$ and $[-1, 1]$ are the same as those of $[-2, 1]$ and $[-1, 2]$, i.e.

$$\delta_H([-2, 2], [-1, 1]) = \delta_H([-2, 1], [-1, 2]) = 1,$$

$$\rho_2([-2, 2], [-1, 1]) = \rho_2([-2, 1], [-1, 2]) = 1,$$

although from an intuitive point of view the distance between the second pair should be greater.

In contrast, $d_\lambda([-1, 1], [-2, 2]) = \sqrt{1/3}$ and $d_\lambda([-2, 1], [-1, 2]) = 1$ holds, which is more coherent from the intuitive point of view.

It should be noted that the possibility of considering measures $W \in \mathcal{P}^*_\theta$ giving weights $(\theta/2, 1 - \theta, \theta/2)$ to the points $(0, 1/2, 1)$, without loss of generality, allows us to obtain a third alternative expression for $d_W$ in terms of $\theta$

$$d^\theta_W(A, B) = \frac{\theta}{2}(\bar{a} - \bar{b})^2 + \frac{\theta}{2}(a - b)^2 + (1 - \theta)\left(\frac{1}{2}(a - \bar{b}) + \frac{1}{2}(\bar{a} - \bar{b})\right)^2$$ \hspace{1cm} (10)

Although the distance of two intervals with respect to the Hausdorff metric $\delta_H$ and the metric $d_W$ may be completely different, both metrics induce the same notion of convergence.
Proposition 3.1 For every probability measure $W \in \mathcal{P}_0^*$ we have that

$$\delta_H(A, B) \sqrt{\frac{\theta_W}{2}} \leq d_W(A, B) \leq \delta_H(A, B)$$

(11)

for all $A, B \in \mathcal{K}_c(\mathbb{R})$, and hence the metric $d_W$ is topologically equivalent to the Hausdorff metric $\delta_H$ on $\mathcal{K}_c(\mathbb{R})$.

Proof: The inequalities are obtained from (7) and (10).

4 Extension to $\mathcal{K}_c(\mathbb{R}^d)$

Since the metric $d_W$ on $\mathcal{K}_c(\mathbb{R})$ can be expressed in terms of mids and spreads (9) we first introduce some generalized concepts of mid and spread in order to extend this interesting metric to the multidimensional setting $\mathcal{K}_c(\mathbb{R}^d)$.

4.1 Generalized mid and spread

In many situations, the spread of an interval is connected with the imprecision (i.e., the smaller the spread is, the closer the interval is to a precise real number). In the multidimensional case, the imprecision can vary depending on the direction. A way for generalizing the concepts of mid and spread is to consider the set of all different directions (which can be done by the unit sphere $S^{d-1}$), and for a given $A \in \mathcal{K}_c(\mathbb{R}^d)$ calculate the lengths $\pi_u(A)$ of all orthogonal projections of $A$ on the direction $u$ for every $u \in S^{d-1}$, i.e.

$$\pi_u(A) = [\pi_u(A), \pi_u(A)] = [-s_A(-u), s_A(u)].$$

(12)

Having this we define $\text{mid}_A, \text{spr}_A : S^{d-1} \to \mathbb{R}$ by

$$\text{mid}_A(u) = \frac{1}{2}(s_A(u) - s_A(-u)), \quad \text{spr}_A(u) = \frac{1}{2}(s_A(u) + s_A(-u)).$$

(13)

Remark 4.1 In case of $d = 1$, $S^{d-1} = \{-1, 1\}$ and $\text{mid}_A(-1) = -\text{mid}_A$, $\text{mid}_A(1) = \text{mid}_A$, $\text{spr}_A(-1) = \text{spr}_A(1) = \text{spr}_A$ holds.

The functions $\text{mid}_A$ and $\text{spr}_A$ fulfill the following properties.

Proposition 4.1 Let $A, B \in \mathcal{K}_c(\mathbb{R}^d)$ and $r \in \mathbb{R}$, then

(1) $\text{mid}_A$ is an odd function, i.e., $\text{mid}_A(-u) = -\text{mid}_A(u)$ for all $u \in S^{d-1}$.
(2) $\text{spr}_A$ is an even function, i.e., $\text{spr}_A(-u) = \text{spr}_A(u)$ for all $u \in S^{d-1}$. 
Both \( \text{mid}_A \) and \( \text{spr}_A \) are Lipschitz-continuous (Lipschitz constant depending on \( A \)).

(4) \( \text{mid} ( \cdot ) \) is linear, i.e., \( \text{mid}_A + B = \text{mid}_A + \text{mid}_B \) and \( \text{mid}_{rA} = r \text{mid}_A \).

(5) \( \text{spr} ( \cdot ) \) is semilinear, i.e., \( \text{spr}_A + B = \text{spr}_A + \text{spr}_B \) and \( \text{spr}_{rA} = |r| \text{spr}_A \).

(6) The support function can be decomposed in the sum of \( \text{mid} \) and \( \text{spr} \), i.e.

\[
\text{s}_A = \text{mid}_A + \text{spr}_A
\]

\[
\text{spr}_A \text{ is the support function of the convex compact set } \left( \frac{1}{2} A + \frac{1}{2} (-A) \right),
\]

having \( \text{mid} \) identically 0, i.e.,

\[
\text{spr}_A = \text{s} \left( \frac{1}{2} A + \frac{1}{2} (-A) \right) \text{ and } \text{mid}_A \left( \frac{1}{2} A + \frac{1}{2} (-A) \right) = 0.
\]

(8) \( \int_{S^{d-1}} \text{mid}_A (u) d\vartheta (u) = 0, \quad \int_{S^{d-1}} \text{spr}_A (u) d\vartheta (u) = 0 \)

We propose to extend the metric \( d_W \) to \( K_c (\mathbb{R}^d) \) via the generalized \( \text{mid} \) and spread by mimicking (9), whereby the Euclidean distance in \( \mathbb{R} \) is replaced by the usual \( L_2 \)-distance \( \| \cdot \|_2 \) for functions defined on \( S^{d-1} \) with respect to \( \vartheta_d \), and the relative importance of the spreads against the mids is given by means of a parameter \( \theta \in (0, 1] \), that is,

\[
d_{\theta, 2} (A, B) = \| \text{mid}_A - \text{mid}_B \|_2^2 + \theta \| \text{spr}_A - \text{spr}_B \|_2^2.
\] (14)

In the next theorem we present alternative expressions for \( d_{\theta, 2} \) that are very useful both from a theoretical and an intuitive point of view. The first one links (14) with \( d_W \) by considering the projections on each direction. The second one allows us to express it the squared new metric \( d_{\theta, 2} \) as a convex combination of the usual \( L_2 \)-metric \( \rho_2^2 (A, B) \) and an extra term generated by the \( L_2 \)-metric of two combinations of the sets \( A, B \).

**Theorem 4.1** Let \( \theta \in (0, 1] \), then for any \( W \in \mathcal{P}_0 \) such that \( \theta_W = \theta \) it follows that

\[
d_{\theta_W, 2} (A, B) = \int_{S^{d-1}} d_{W}^2 \left( \pi_u (A), \pi_u (B) \right) d\vartheta_d (u)
\]

\[
= \theta \rho_2^2 (A, B) + (1 - \theta) \rho_2^2 \left( -\frac{1}{2} A + \frac{1}{2} B , -\frac{1}{2} B + \frac{1}{2} A \right)
\] (15) (16)

holds for every pair \( A, B \in K_c (\mathbb{R}^d) \).

**Proof:** First of all note that (10) and (12) imply that

\[
\text{...}
\]
\[
\int_{S^{d-1}} d_W^2(\pi_u(A), \pi_u(B)) d\vartheta_d(u) = \frac{\theta}{2} \int_{S^{d-1}} (s_A(u) - s_B(u))^2 d\vartheta_d(u) \\
+ \frac{\theta}{2} \int_{S^{d-1}} (s_B(-u) - s_A(-u))^2 d\vartheta_d(u) \\
+ (1 - \theta) \int_{S^{d-1}} \left( \frac{1}{2}(s_B(-u) - s_A(-u)) + \frac{1}{2}(s_A(u) - s_B(u)) \right)^2 d\vartheta_d(u) \\
= \theta \rho_2^2(A, B) + (1 - \theta) \rho_2^2(-\frac{1}{2}A + \frac{1}{2}B, -\frac{1}{2}B + \frac{1}{2}A).
\]

On the other hand, a straightforward calculation shows that
\[
\rho_2^2(A, B) = \|\text{mid}_A - \text{mid}_B\|^2 + \|\text{spr}_A - \text{spr}_B\|^2. \tag{17}
\]

From (17) and Proposition 4.1 it finally follows that
\[
\rho_2^2\left(-\frac{1}{2}A + \frac{1}{2}B, -\frac{1}{2}B + \frac{1}{2}A\right) = \|\text{mid}_A - \text{mid}_B\|^2.
\]
which completes the proof. \quad \Box

Let \( \theta \in (0, 1) \) and consider \( W \in \mathcal{P}_0 \) such that \( \theta_W = \theta \). From Remark 4.1 it follows that in case of \( d = 1 \)
\[
d_{\theta, 2}^2(A, B) = (\text{mid}_A - \text{mid}_B)^2 + \theta(\text{spr}_A - \text{spr}_B)^2 = \frac{1}{2}d_W^2(A, B) + \frac{1}{2}d_W^2(-A, -B).
\]
If \( W \in \mathcal{P}_0^+ \) then \( d_W^2(A, B) = d_W^2(-A, -B) \) and consequently \( d_{\theta, 2} \) is equal to the metric by Bertoluzza et al. [2]. This means that \( d_{\theta, 2} \) is a kind of correction of \( d_W \) in order to get a metric invariant to isometries (see Remark 3.1).

In the next theorem we prove that \( d_{\theta, 2} \) indeed is a metric.

**Theorem 4.2** For every \( \theta \in (0, 1] \) the mapping \( d_{\theta, 2} : \mathcal{K}_c(\mathbb{R}^d) \times \mathcal{K}_c(\mathbb{R}^d) \to [0, \infty) \), defined according to (14), is a metric.

**Proof:** Let \( W \in \mathcal{P}_0 \) such that \( \theta_W = \theta \). Given \( A, B \in \mathcal{K}_c(\mathbb{R}^d) \) define \( f_{AB} : S^{d-1} \to [0, \infty) \) by \( f_{AB}(u) = d_W\left(\pi_u(A), \pi_u(B)\right) \). It follows from the properties of support functions [19] and from (10) that \( f_{AB} \) is (uniformly) continuous on \( S^{d-1} \) and therefore Borel-measurable.

Since \( d_{\theta, 2}(A, B) = \|f_{AB}\|_2 \) holds for every \( A, B \in \mathcal{K}_c(\mathbb{R}^d) \) and since \( f_{AB}(u) \leq f_{AC}(u) + f_{CB}(u) \) is obviously fulfilled for every \( A, B, C \in \mathcal{K}_c(\mathbb{R}^d) \) it follows immediately that \( d_{\theta, 2} \) fulfills the triangle inequality and that \( d_{\theta, 2}(\cdot, \cdot) \geq 0 \). Furthermore \( d_{\theta, 2} \) is clearly symmetric. It remains to verify that \( d_{\theta, 2}(A, B) = 0 \) implies \( A = B \). In this sense, if \( d_{\theta, 2}(A, B) = 0 \) then \( d_W\left(\pi_u(A), \pi_u(B)\right) = 0 \) holds for \( \vartheta \)-almost every \( u \in S^{d-1} \), which in particular implies that \( s_A = s_B \).
on a dense subset of $S^{d-1}$. Using (Lipschitz-) continuity of support functions completes the proof.

In order to show the good intuitive meaning of $d_\theta$ illustrated in Example 3.1 is also fulfilled by $d_{\theta,2}$ on $K_c(\mathbb{R}^d)$ we consider the following example.

**Example 4.1** As in Example 3.1, we consider $\theta = 1/3$ (the mass obtained for the Lebesgue measure). We will calculate the distances $d_{1/3,2}(A_1, B_1)$ and $d_{1/3,2}(A_2, B_2)$ for the convex compact sets $A_1 = K((0, 0), 1), B_1 = K((0, 0), 2), A_2 = K((-1/2, 0), 3/2)$ and $B_2 = K((1/2, 0), 3/2)$, whereby $K((x, y), r)$ denotes the closed ball with center $(x, y)$ and radius $r$ in $\mathbb{R}^2$ (see Figure 1).

First of all, it is easy to check that $\text{mid}_K((0, 0), r)(u) = 0$ and $\text{spr}_K((0, 0), r)(u) = r$ for all $u \in S^1$. Consequently, by (14) we have that $d_{1/3,2}^2(A_1, B_1) = 1/3$.

On the other hand, from Proposition 4.1 we have that $\text{mid}_{K((x, y), r)}(u) = \text{mid}_{\{(x, y)\}}(u)$ and $\text{spr}_{K((x, y), r)}(u) = r$ for all $u \in S^1 \text{ and all } (x, y) \in \mathbb{R}^2$. Thus,

$$d_{1/3,2}^2(A_2, B_2) = \|\text{mid}_{\{-1/2,0\}} - \text{mid}_{\{(1/2,0)\}}\|_2^2 + 0 = \|\text{mid}_{\{(1,0)\}}\|_2^2$$

$$= \frac{1}{2\pi} \int_{0,2\pi} \cos^2(\varphi)d\varphi = \frac{1}{2}.$$
In particular the inequality \( d_{1/3}^2(A_2, B_2) = 1/2 > 1/3 = d_{1/3}^2(A_1, B_1) \) holds. The corresponding Hausdorff distance is \( \delta_H(A_1, B_1) = \delta_H(A_2, B_2) = 1 \), and the usual \( L^2 \)-metric is \( \rho_2^2(A_1, B_1) = 1 \) and \( \rho_2^2(A_2, B_2) = 1/2 \) respectively. Thus, an analogous drawback that the one remarked in [2] comes up in \( K_c(\mathbb{R}^d) \) in connection with \( \delta_H \) and \( \rho_2 \), meanwhile this does not happen for \( d_{\theta,2} \).

The following result concerns the equality of the topologies generated by \( d_{\theta,2} \) and \( \delta_H \) - this is the multidimensional version of Proposition 3.1.

**Proposition 4.2** For every \( \theta \in (0, 1] \) we have that

1. \( d_{\theta,2}(A, B) \leq \delta_H(A, B) \) and
2. \( d_{\theta,2}(A, B) \geq \sqrt{\theta} c(d, \text{diam}(A \cup B)) \left( \delta_H(A, B) \right)^{\frac{\theta+1}{2}} \)

for all \( A, B \in K_c(\mathbb{R}^d) \), where \( c(d, \text{diam}(A \cup B)) \) denotes the well-known constant due to Vitale [20]. As a result, the metric \( d_{\theta,2} \) induces the same topology as the Hausdorff-metric \( \delta_H \) on \( K_c(\mathbb{R}^d) \).

**Proof:** From the properties of the Hausdorff metric \( \delta_H \) and equation (16) it follows that

\[
d_{\theta,2}(A, B) \leq (1 - \theta) \delta_H^2 \left( -\frac{1}{2}B + \frac{1}{2}A, -\frac{1}{2}A + \frac{1}{2}B \right) + \theta \delta_H^2(A, B) \\
\leq (1 - \theta) \left( \frac{1}{2} \delta_H(B, A) + \frac{1}{2} \delta_H(A, B) \right)^2 + \theta \delta_H^2(A, B) \\
= \delta_H^2(A, B),
\]

which proves (1). Inequality (2) is a direct consequence of the analogous inequality in [20].

From the mathematical point of view the metric space \( (K_c(\mathbb{R}^d), d_{\theta,2}) \) has the same properties as \( (K_c(\mathbb{R}^d), \delta_H) \).

**Proposition 4.3** For every \( \theta \in (0, 1] \) \( (K_c(\mathbb{R}^d), d_{\theta,2}) \) is a complete, separable metric space, in which every closed, bounded subset is compact.

**Proof:** The assertion can be proved in the same way as proving that \( (K_c(\mathbb{R}^d), \rho_2) \) is a complete, separable metric space, in which every closed, bounded subset is compact [20].

Furthermore, as it is stated in the next proposition, \( d_{\theta,2} \) fulfills the important invariance property mentioned in Remark 3.1.
Proposition 4.4 Let $\theta \in (0, 1]$ then $d_{\theta, 2}$ is invariant to with respect to any isometry $T : \mathbb{R}^d \to \mathbb{R}^d$, i.e. $d_{\theta, 2}(T(A), T(B)) = d_{\theta, 2}(A, B)$ holds for every pair $A, B \in \mathcal{K}_c(\mathbb{R}^d)$.

Proof: It is well known that every isometry $T$ on $\mathbb{R}^d$ can be written as $T x = Mx + a$, with an orthogonal matrix $M$ and a vector $a \in \mathbb{R}^d$.

Using Proposition 4.1 it is easy to verify that $d_{\theta, 2}$ is invariant to translations. On the other hand, if $M$ is an orthogonal matrix, then $s_{M(A)}(u) = s_A(M^{-1}u)$ and consequently $\mid \text{mid}_{M(A)}(u) = \mid \text{mid}_A(M^{-1}u)$ and $\text{spr}_{M(A)}(u) = \text{spr}_A(M^{-1}u)$ holds for every $u \in S^{d-1}$ and every $A \in \mathcal{K}_c(\mathbb{R}^d)$. Consequently, using transformation rules (see [7]) of the Lebesgue integral ($|\text{det}(M)| = 1$), it follows immediately that

$$\|\text{mid}_{M(A)} - \text{mid}_{M(B)}\|_2 = \|\text{mid}_A - \text{mid}_B\|_2$$

$$\|\text{spr}_{M(A)} - \text{spr}_{M(B)}\|_2 = \|\text{spr}_A - \text{spr}_B\|_2$$

which completes the proof. $\square$

5 Extension to $\mathcal{F}_c(\mathbb{R}^d)$

The extension of the metric $d_{\theta, 2}$ from $\mathcal{K}_c(\mathbb{R}^d)$ to $\mathcal{F}_c(\mathbb{R}^d)$ can be done as outlined in the introduction, that is, by considering a weight probability measure $\varphi$ on $[0, 1]$ and by defining

$$D^\varphi_{\theta, 2}(\tilde{A}, \tilde{B}) = \left( \int_{[0, 1]} d^2_{\theta, 2}(\tilde{A}_{\alpha}, \tilde{B}_{\alpha})d\varphi(\alpha) \right)^{1/2}$$

(18)

To assure that (18) really yields a metric the weight measure $\varphi$ must have the full interval as support, i.e. its distribution function should be strictly increasing. The fact that under this assumption $D^\varphi_{\theta, 2}$ satisfies the conditions of metric on $\mathcal{F}_c(\mathbb{R}^d)$ can be proved analogous to Lemma 4.2. Note that good theoretical and intuitive properties of this metric are directly inherited from those established in the previous sections.

From the practical point of view one may choose the weight measure $\varphi$ in such a way that $\varphi$ reflects the intuitive (or subjective) interpretation of fuzzy sets - for instance one may treat every $\alpha$-level as equally important (and therefore use the Lebesgue measure as weight measure $\varphi$) or give more mass to $\alpha$-levels close to 1.

Choosing the Lebesgue measure $\lambda$ as weight measure induces a notion of convergence that is equivalent to that induced by the usual $\rho_2$ metric (6) on $\mathcal{F}_c(\mathbb{R}^d)$. 

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Proposition 5.1 For every $\theta \in (0, 1]$ the metric $D_{\theta,2}^\lambda$ generates the same topology on $\mathcal{F}_c(\mathbb{R}^d)$ as the metric $\rho_2$, defined according to (6). In particular, $(\mathcal{F}_c(\mathbb{R}^d), D_{\theta,2}^\lambda)$ is a separable metric space.

The concept of mid and spread in $\mathcal{K}_c(\mathbb{R}^d)$ in (13) can be level-wise extended to $\mathcal{F}_c(\mathbb{R}^d)$, that is,

$$\text{mid} \tilde{\alpha} (u, \alpha) = \text{mid} \tilde{\alpha}_\alpha (u), \quad \text{spr} \tilde{\alpha} (u, \alpha) = \text{spr} \tilde{\alpha}_\alpha (u)$$

for all $u \in S^{d-1}$ and $\alpha \in [0, 1]$. Consequently, as before in the crisp case, we get the following expression ($\| \cdot \|_2$ denoting the usual $L_2$-norm on the Hilbert space $\mathcal{H} = L^2(S^{d-1} \times [0, 1], \theta_d \otimes \lambda)$)

$$\left(D_{\theta,2}^{\varphi} (\tilde{A}, \tilde{B}) \right)^2 = \| \text{mid} \tilde{A} - \text{mid} \tilde{B} \|^2_2 + \theta \| \text{spr} \tilde{A} - \text{spr} \tilde{B} \|^2_2$$

As before we observe that according to (20) the parameter $\theta$ plays the role of a relative weight of the distance of the generalized spread against the distance of the generalized mid.

6 Concluding Remarks

We have defined a family of metrics for fuzzy sets with convex compact $\alpha$-level sets on the basis of a suitable family of metrics on $\mathcal{K}_c(\mathbb{R}^d)$ that was motivated by the paper of Bertoluzza et al.[2]. These metrics exhibit very good mathematical properties in addition to having an adequate intuitive interpretation and moreover they are easy to compute in terms of generalized mid and spread functions.

The concept of generalized mid and spread may be very useful in working with (fuzzy) convex compact sets, because the functions are easy to handle and strongly related to the widely used support functions.

The metrics considered in this work are $L_2$-type distances, and consequently they are very useful in the statistical context. For this reason this kind of metrics can successfully be used in the context of working with fuzzy random variables (see for instance, [5,9]).

In the future it would be interesting to deeply analyze the effect of the choice of $\theta$ and $\varphi$ both in practical situations as well as from a theoretical point of view. In particular it would be interesting to observe under which general conditions the metrics $D_{\theta,2}^{\varphi_1}$ and $D_{\theta,2}^{\varphi_2}$ induce the same notion of convergence (i.e. the same topology).
References


