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ON PARAMETER ESTIMATION FOR THE THREE
PARAMETER WEIBULL DISTRIBUTION AND
ESTIMATION OF THE RELIABILITY FUNCTION BASED
ON FUZZY LIFE TIME DATA

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Abstract
Exact measurements of continuous variables are not possible. The observed
data always have some imprecision but standard statistical tools do not con-
sider this imprecision. To cover this fuzziness Zadeh introduced the fuzzy
set concept. Since life time observations are also more or less fuzzy, this
study is conducted to form the characterizing functions of the estimates of
the parameters of the three parameter Weibull distribution based on fuzzy
life time data. Moreover a generalized estimate for the reliability function is
given.

Key Words: Characterizing function; Non-precise data; Support; δ-cut
1 Introduction

In the world of real measurements continuous quantities cannot be measured exactly. The precision (uncertainty) of a measurement depends on the accuracy of the measuring instrument. Hence it is impractical to find out an instrument which exactly measures the unit, for instance quantities like age of a person, height of a person, life time of equipment, intensity of light, water level of river. This shows that a real measurement always results in a non-precise number. Standard statistical models are developed to explain the variability among the observations without considering the fuzziness in single observations. Consequently another method of modeling was necessary to consider the imprecision of a single measurement. To overcome this problem the idea of fuzzy sets was first introduced by Zadeh (Viertl, 2011), (Wu, 2009).

In classical set theory a two valued characteristic function, also called indicator function, is used to represent whether an element \( x \) is in a subset \( A \) of a universal set \( M \) or not, as mentioned in equation (1):

\[
\mathbb{1}_A(x) = \begin{cases} 
1 & \text{if } x \in A \\
0 & \text{if } x \notin A
\end{cases} \quad \forall \ x \in M \tag{1}
\]

Since fuzzy set theory is the extension of classical set theory, i.e. two-valued logic is changed to multi-valued logic, therefore the classical set notations
were also extended to fuzzy set notations. The characteristic function men-
tioned in equation (1) is extended to the so-called membership function $\mu_A$
of a fuzzy subset $A$ of $M$ i.e. equation (2):

$$
\mu_A(x) = \begin{cases} 
1 & \text{if } x \in A \\
\delta & \text{if } x \text{ belongs to } A \text{ to some degree } \delta \\
0 & \text{if } x \notin A
\end{cases}
$$

(2)

The membership function maps the elements from a universal set $M$ to
the interval $[0; 1]$ (Szeliga, 2004).

Standard statistical inference takes the observed values as precise numbers
or vectors. Only few papers considered data in interval form. The mea-
surements from continuous quantities cannot be precise numbers, but they
are more or less imprecise numbers (Viertl, 2006). The most frequently used
term in statistics is stochastic quantity (random variable). But in most of the
situations the observed data have two types of uncertainty, one is stochastic
that can be modeled by probability distributions, and the other is fuzziness
which should be modeled by fuzzy numbers (Viertl, 2011).

**Fuzzy Numbers**

So-called fuzzy numbers $x^*$ are special fuzzy sets determined by the so-called
characterizing function $\xi(\cdot)$ which is a real function of one real variable sat-
isfying the following conditions 1-3:
1. $\xi : \mathbb{R} \rightarrow [0 ; 1]$. 

2. For all $\delta \in (0 ; 1]$ the so-called $\delta$-cut $C_{\delta}(x^*) := \{ x \in \mathbb{R} : \xi(x) \geq \delta \}$ is a finite union of compact intervals $[a_{\delta,j} ; b_{\delta,j}]$, i.e.

$$C_{\delta}(x^*) = \bigcup_{j=1}^{k_{\delta}} [a_{\delta,j} ; b_{\delta,j}] \neq \emptyset$$

3. $\xi(\cdot)$ has bounded support, i.e. $\text{supp}[\xi(\cdot)] := \{ x \in \mathbb{R} : \xi(x) > 0 \} \subseteq [a ; b]$

The set of all fuzzy numbers are represented by $\mathcal{F}(\mathbb{R})$.

If all $\delta$-cuts of a fuzzy number are non-empty closed bounded intervals, the corresponding fuzzy number is called fuzzy interval.

**Remark:** The family $C_{\delta}(x^*) ; \delta \in (0 ; 1]$ is nested, i.e. $\delta_1 < \delta_2$ we have $C_{\delta_1} \supseteq C_{\delta_2}$

**Fuzzy Vectors**

A $n$-dimensional fuzzy vector $x^*$ is determined by its so-called vector characterizing function $\zeta(\cdot, \ldots, \cdot)$ which is a real function of $n$ real variables $x_1, x_2, \ldots, x_n$ obeying the following three conditions:

1. $\zeta : \mathbb{R}^n \rightarrow [0 ; 1]$.

2. For all $\delta \in (0 ; 1]$ the so-called $\delta$-cut $C_{\delta}[x^*] := \{ x \in \mathbb{R}^n : \zeta(x) \geq \delta \}$ is non-empty, bounded, and a finite union of simply connected and closed sets.
3. The support of $\zeta(\cdot, \ldots, \cdot)$ is a bounded set.

The set of all $n$-dimensional fuzzy vectors is denoted by $\mathcal{F}(\mathbb{R}^n)$.

Remark: The family $(C_\delta(x^*); \delta \in (0; 1])$ is nested, i.e. for $\delta_1 < \delta_2$ we have $C_{\delta_1}(x^*) \supseteq C_{\delta_2}(x^*)$.

From the above mentioned information it is important to note that the characterizing function may be obtained from the family of $(C_\delta(x^*); \delta \in (0; 1])$ as described below:

Lemma: Denoting by $1_A(\cdot)$ the indicator function of the set $A \subseteq \mathbb{R}$, for any characterizing function $\xi(\cdot)$ of a fuzzy number $x^*$ the following is valid:

$$\xi(t) = \max\left\{\delta \cdot 1_{C_\delta(x^*)}(x) : \delta \in [0; 1]\right\} \forall x \in \mathbb{R} \ (\text{Viertl, 2011}).$$

Remark: It should be noted that not all families $(A_\delta; \delta \in (0; 1])$ of nested finite unions of compact intervals are the $\delta$-cuts of a fuzzy number. But the following construction lemma holds:

Construction Lemma: Let $(A_\delta; \delta \in (0; 1])$ with $A_\delta = \bigcup_{j=1}^{k_\delta} [a_{\delta,j}; b_{\delta,j}]$ be a nested family of non-empty subsets of $\mathbb{R}$. Then the characterizing function $\xi(\cdot)$ of the generated fuzzy number $x^*$ is given by

$$\xi(t) := \sup\left\{\delta \cdot 1_{A_\delta}(x) : \delta \in [0; 1]\right\} \forall x \in \mathbb{R} \ (\text{Viertl and Hareter, 2006}).$$
Extension Principle:

This is the generalization of an arbitrary function $g: M \rightarrow N$ for fuzzy argument value $a^*$ in $M$. Let $a^*$ be a fuzzy element of $M$ with membership function $\mu: M \rightarrow [0; 1]$, then the fuzzy value $y^* = g(a^*)$ is the fuzzy element $y^*$ in $N$ whose membership function $\nu(\cdot)$ is defined by

$$\nu(y) := \begin{cases} 
\sup \{ \mu(a) : a \in M, \ g(a) = y \} & \text{if } \exists a : g(a) = y \\
0 & \text{if } \not\exists a : g(a) = y 
\end{cases} \forall y \in N$$

(Klir and Yuan, 1995).

Theorem: For a continuous function $f: \mathbb{R} \rightarrow \mathbb{R}$, and for a fuzzy interval $x^*$ applying the extension principle the following holds true: following holds true:

$$C_\delta [f(x^*)] = \left[ \min_{x \in C_\delta(x^*)} f(x) ; \max_{x \in C_\delta(x^*)} f(x) \right] \forall \delta \in (0 ; 1]$$

where $\min_{x \in C_\delta(x^*)} f(x)$ is lower end of the $\delta$-cut of $f(x^*)$ and $\max_{x \in C_\delta(x^*)} f(x)$ is upper end of the $\delta$-cut of $f(x^*)$ (Viertl, 2011).

If we have an observed classical sample $x_1, x_2, ..., x_n$ of a stochastic quantity $X$ then $x_i$ are elements of the observation space $M_X$ of $X$ and $(x_1, x_2, ..., x_n)$ is an element of the Cartesian product $M_X \times M_X \times ... \times M_X$ of $n$ copies of $M_X$. This makes the sample space of $X$, denoted by $M^n_X$.

But dealing with fuzzy observations $x^*_1, x^*_2, ..., x^*_n$ we cannot do the same be-
cause for fuzzy observations each $x_i^*$ is a fuzzy element of $M_X$ but $(x_1^*, x_2^*, ..., x_n^*)$

is not a fuzzy element of the sample space $M^n_X$.

To obtain a fuzzy element of the sample space $M^n_X$ from a fuzzy sample $x_1^*, x_2^*, ..., x_n^*$ with corresponding characterizing functions $\xi_1(\cdot), \xi_2(\cdot), ..., \xi_n(\cdot)$ respectively, from fuzzy theory the minimum t-norm is applied.

The vector characterizing function $\zeta(., ..., .)$ of the combined fuzzy sample $x^*$ applying the minimum t-norm i.e.

$$\zeta(x_1, x_2, ..., x_n) = \min \{\xi_1(\cdot), \xi_2(\cdot), ..., \xi_n(\cdot)\} \quad \forall (x_1, x_2, ..., x_n) \in \mathbb{R}^n.$$ 

Let $x_1^*, x_2^*, ..., x_n^*$ be a fuzzy random sample with characterizing functions $\xi_1(\cdot), \xi_2(\cdot), ..., \xi_n(\cdot)$ respectively for all $\delta$-cuts $C_\delta(x_i^*) i = 1(1)n$, the $\delta$-cuts of the combined fuzzy sample will be

$$C_\delta [\zeta(., ..., .)] = \times_{i=1}^n C_\delta [\xi_i(\cdot)] \quad \forall \delta \in (0 ; 1]$$ (Viertl, 2006).

It is a fact that the lifetime of any system, component or individual is random and unpredictable, and hence it is amenable to statistical laws.

Therefore in the twentieth century the development started to make statistical models for life times. In life time analysis the variable of interest (life time) is the time until the occurrence of a specified event. Mostly the event is death in biological phenomena, or failure in the engineering field. Main aim of these analysis are predicting the probability of an event, survival time or reliability (Deshpande and Purohit, 2005); (Lee and Wang, 2013)).

As discussed earlier continuous phenomena cannot be measured accu-
rately, therefore lifetime observations are also non-precise numbers, i.e. they
are more or less fuzzy (Viertl, 2009).

2 Generalized Estimation for the Weibull Distribution

The Weibull distribution is one of the most popular models for life times. The
density \( f(x|\alpha, \beta, \gamma) \) and the reliability function \( R(\cdot) \) of the three parameter
Weibull distribution are given by

\[
f(x|\alpha, \beta, \gamma) = \frac{\gamma}{\beta} \left( \frac{x - \alpha}{\beta} \right)^{\gamma-1} \exp \left\{ - \left( \frac{x - \alpha}{\beta} \right)^\gamma \right\} \mathbb{1}_{(\alpha, \infty)}(x) \quad \forall \, x > 0.
\]

\( \alpha \): Location Parameter

\( \beta \): Scale Parameter

\( \gamma \): Shape Parameter

And the three parameter Weibull reliability function is

\[
R(x) = \exp \left\{ - \left( \frac{x - \alpha}{\beta} \right)^\gamma \right\} \quad \forall \, x \geq \alpha.
\]
Various studies have been conducted on the parameter estimation of the three parameter Weibull distribution like Maximum Likelihood Estimator (Wang et al., 2011), Least Squares Method (Soman and Misra, 1992), Modified Weighted Least Squares Estimators (Ahmad, 1994), and Moment Estimators for the 3-Parameter Weibull Distribution (Cran, 1988). All of the mentioned techniques are based on observed life times in form of precise numbers.

Research has been conducted for estimation of the parameters of the two parameter Weibull distribution (Hung and Liu, 2004), reliability estimation (Viertl, 2009), and Bayesian reliability estimation (Huang et al., 2006) using fuzzy data.

For the best of our knowledge the three parameter Weibull distribution estimation was not performed for fuzzy data. Denoting by \( \nu(\ldots, \ldots, \ldots) \) a classical estimator of the corresponding Weibull parameter \( \theta \) of the three parameter Weibull distribution for precise observations, i.e. \( \nu(x) = \hat{\theta} \) the parameters \( \alpha, \beta, \) and \( \gamma \) can be estimated by

\[
\hat{\alpha} = \frac{m_1 \cdot m_4 - m_2^2}{m_1 + m_4 - 2m_2}
\]

\[
\hat{\gamma} = \frac{\ln 2}{\ln(m_1 - m_2) - \ln(m_2 - m_4)}
\]
\[
\hat{\beta} = \frac{\bar{m}_1 - \hat{\alpha}}{\Gamma\left(1 + \frac{1}{\hat{\gamma}}\right)}
\]

where
\[
\bar{m}_k = \sum_{r=0}^{n-1} \left(1 - \frac{r}{n}\right)^k \left(x_{(r+1)} - x_{(r)}\right) \quad x_{(0)} = 0
\]

Now if \(\hat{\alpha}^*\) is a fuzzy parameter estimator for the location parameter of the three parameter Weibull distribution based on fuzzy data then its \(\delta\)-cuts are denoted by
\[
C_\delta(\hat{\alpha}^*) = [\hat{\alpha}^*_L ; \hat{\alpha}^*_U] \quad \forall \delta \in (0 ; 1],
\]

where
\[
\hat{\alpha}^*_L = \min_{x \in \times_{i=1}^n C_\delta(x^*_i)} \frac{\bar{m}_1 \cdot \bar{m}_4 - \bar{m}_2^2}{\bar{m}_1 + \bar{m}_4 - 2\bar{m}_2}
\]

and
\[
\hat{\alpha}^*_U = \max_{x \in \times_{i=1}^n C_\delta(x^*_i)} \frac{\bar{m}_1 \cdot \bar{m}_4 - \bar{m}_2^2}{\bar{m}_1 + \bar{m}_4 - 2\bar{m}_2}
\]

In a similar way \(\hat{\beta}^*, \hat{\gamma}^*\) fuzzy estimators of scale and shape parameter of the three parameter Weibull distribution are constructed respectively. Its \(\delta\)-cuts can also be defined by
\[
C_\delta\left(\hat{\beta}^*\right) = [\hat{\beta}^*_L ; \hat{\beta}^*_U] \quad \forall \delta \in (0 ; 1],
\]
\[ \hat{\beta}_\delta^L = \min_{x \in \times_{i=1}^{\infty} C_{\delta}(x^*_i)} \frac{m_1 - \hat{\alpha}}{\Gamma \left(1 + \frac{1}{\gamma}\right)} \]

and

\[ \hat{\beta}_\delta^U = \max_{x \in \times_{i=1}^{\infty} C_{\delta}(x^*_i)} \frac{m_1 - \hat{\alpha}}{\Gamma \left(1 + \frac{1}{\gamma}\right)} \]

where

\[ C_{\delta}(\hat{\gamma}^*) = \left[ \hat{\gamma}_\delta^L, \hat{\gamma}_\delta^U \right] \quad \forall \delta \in (0; 1], \]

\[ \hat{\gamma}_\delta^L = \min_{x \in \times_{i=1}^{\infty} C_{\delta}(x^*_i)} \frac{\ln2}{\ln(m_1 - m_2) - \ln(m_2 - m_4)} \]

and

\[ \hat{\gamma}_\delta^U = \max_{x \in \times_{i=1}^{\infty} C_{\delta}(x^*_i)} \frac{\ln2}{\ln(m_1 - m_2) - \ln(m_2 - m_4)} \]

As an example take fuzzy life time observations with trapezoidal characterizing functions as given in figure 1.

Figure 1: Trapezoidal fuzzy life times

\[ \xi_i(t) \]
For the construction of the characterizing function $\psi(\cdot)$ of the fuzzy estimates of the Weibull parameters the following steps are applied:

1. The values for $\delta$ are taken from 0 to 1 with an increment 0.1.

2. For a given value of $\delta$ all $\delta$-cuts of the fuzzy observations are determined.

3. Taking 10 values from all $\delta$-cuts (for $n$ observations we obtain $10^n$ values) we obtain hypothetical classical samples.

4. From these hypothetical classical samples at a given level $\delta$, the standard classical estimates of the parameters are calculated.

5. In order to construct the characterizing function of a generalized (fuzzy) estimator $\hat{\theta}^*$ the minimum and maximum values from these estimates are taken and are considered as the end points of the family $(A_\delta; \delta \in (0; 1])$ of generating intervals $A_\delta$ of the characterizing function of the fuzzy estimators at a given level of $\delta$.

6. Steps 3-5 are performed for each for $\delta = 0(0.1)1$.

7. From all these generating intervals $A_\delta$ obtained for each $\delta$ (i.e. $\delta = 0^+, 0.1, 0.2, ..., 1$) through the above mentioned Construction Lemma the characterizing functions of the fuzzy estimates of the parameters are obtained.

In figures 2 - 4 the characterizing functions of the fuzzy estimates of the Weibull parameters $\alpha$, $\beta$, and $\gamma$ based on the fuzzy sample from figure 1 are depicted:
Figure 2: Characterizing function of the fuzzy estimate $\hat{\alpha}^*$

$\psi(\alpha)$

Figure 3: Characterizing function of the fuzzy estimate $\hat{\beta}^*$

$\psi(\beta)$
3 Estimation of the Reliability Function

The lower and upper $\delta$-level curves of the fuzzy estimate $R^*(\cdot)$ of the reliability function $R(\cdot)$ of a three parameter Weibull distribution are obtained from the $\delta$-cuts

$$C_{\delta}(R^*(x)) = \left[ \min \left\{ \exp \left\{ - \left( \frac{x - \hat{\alpha}^*}{\hat{\beta}^*} \right)^{\hat{\gamma}^*} \right\} \right\} ; \max \left\{ \exp \left\{ - \left( \frac{x - \hat{\alpha}^*}{\hat{\beta}^*} \right)^{\hat{\gamma}^*} \right\} \right\} \right] \forall \delta \in (0; 1]$$

The fuzzy estimate of the reliability function of the three parameter Weibull distribution is obtained through the following algorithm:

1. The values taken for $\delta$ are 0, 0.5, 1.

2. For a given value of $\delta$ all $\delta$-cuts of the fuzzy observations are determined.
3. Taking 10 values from all $\delta$-cuts (for n observations we obtain $10^n$ values) hypothetical classical samples are obtained.

4. From these hypothetical classical samples at a given level $\delta$, the standard classical estimates of the parameters are calculated.

5. Based on these estimates the values of the reliability function given in equation 2 are calculated for times 110, 120, 130, 140, 150, 160, 170, 180, 190, 200, 212.

6. The lower and upper $\delta$-level curves of the fuzzy estimate of the reliability function are constructed by taking minimum and maximum values of the classical estimated values of the reliability function at time points 110, 120, 130, 140, 150, 160, 170, 180, 190, 200, 212 at a given level of $\delta$.

7. The minimum values make the lower $\delta$-level curve, and the maximum values make the upper $\delta$-level curve.

8. Steps 2-7 are performed for $\delta = 0, 0.5, \text{and} \ 1$. 
Conclusion

Life time is a continuous phenomenon that cannot be measured exactly. Therefore observing the life time we should consider it as fuzzy number, and, instead of standard statistical parameter estimation, fuzzy techniques are more suitable to obtain appropriate and realistic results. In this paper a method to obtain characterizing functions for the estimates of the three parameters of Weibull distributions are constructed. Moreover based on fuzzy samples upper and lower -level curves for the reliability function can be calculated.
References


