On the Equivalence of Quadratic Optimization Problems Commonly Used in Portfolio Theory

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Abstract
In the paper, we consider three quadratic optimization problems which are frequently applied in portfolio theory, i.e., the Markowitz mean-variance problem as well as the problems based on the mean-variance utility function and the quadratic utility. Conditions are derived under which the solutions of these three optimization procedures coincide and are lying on the efficient frontier, the set of mean-variance optimal portfolios. It is shown that the solutions of the Markowitz optimization problem and the quadratic utility problem are not always mean-variance efficient.

The conditions for the mean-variance efficiency of the solutions depend on the unknown parameters of the asset returns. We deal with the problem of parameter uncertainty in detail and derive the probabilities that the estimated solutions of the Markowitz problem and the quadratic utility problem are not mean-variance efficient. Because these probabilities deviate from zero the above mentioned quadratic optimization problems are not stochastically equivalent. The obtained results are illustrated by an empirical study.

The above mentioned quadratic optimization problems, i.e., the Markowitz mean-variance problem and the problems based on the mean-variance utility function and the quadratic utility, are commonly discussed in portfolio theory. The aim of the paper is to compare them with each other and to examine the probability of an unexpected loss under uncertainty. Although their solutions are lying on the same set in the mean-variance space, we show that they are not obviously the same. This topic has only been discussed briefly in recent literature (cf. Leippold et al. (2004), Li and Ng (2000)). Leippold et al. (2004) pointed out that these optimization problems are equivalent from mathematical point of view but not fully equivalent from economical. Here, we extend this result by deriving conditions under which the three optimization procedures are equivalent. These conditions only depend on the parameters of the asset returns. Unfortunately, these parameters, namely the mean vector and the covariance matrix, are usually unknown in practice. As a result, the derived conditions cannot be directly verified in practice since they are functions of these two parameters.

In order to take the parameter uncertainty into account we replace the parameters in the equivalence conditions by their sample estimators. We derive the distributions of these estimated quantities assuming that the asset returns are independently and identically normally distributed. Although the assumption of normality is heavily criticized in financial literature, it can be applied in the case of the mean-variance investor (see Tu and Zhou (2004)). We analyze in detail the probability that the solutions of the optimization problems coincide and derive a test whether a given solution is mean-variance efficient. Contrary to the classical testing theory for the mean-variance efficiency of a portfolio (see e.g. Gibbons et al. (1989), Britten-Jones (1999), Bodnar and Schmid (2009)), the suggested test on efficiency is constructed under the alternative hypothesis and, consequently, it can be accepted.
best of our knowledge such results are not available in literature up to now. Moreover, it is shown how the obtained results can be applied in practice.